Advances in Applied Mathematics 32, 212–238 (2004)
(special issue Workshop on Tutte polynomials, Barcelona 2001)

Bases, Reorientations and Linear Programming in Uniform and Rank 3 Oriented Matroids

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ERRATA

• page 231 line 2: instead of "acting symmetrically on 1457 with three orbits 1457 23689A BCD" read "acting symmetrically on 1357 with three orbits 1357 24689A BCD"

• page 236 figure 7: in the region corresponding to the basis 136, the dark angle should touch the pseudoline 6 instead of the pseudoline 3.

• page 238: add a reference to "R.O. Winder, Partitions of N-spaces by hyperplanes, SIAM J. Applied Math. 14 (1966), 811-818"

In this paper, R.O. Winder proves that the number of regions of an hyperplane arrangement equals $t(2,0)$. Independently of this reference, not well known by combinatoricists, several theorems appeared some years later, from a particular case to a generalization. In 1973, R. Stanley published a combinatorial interpretation of $t(1-\lambda,0)$ in a graph for $\lambda=-1,-2,...$ For $\lambda=-1$, we get that the number of acyclic orientations of a graph equals $t(2,0)$, a particular case of Winder's result. Some generalizations of Stanley's theorem appeared two years later (1975): by T. Zaslavsky in real hyperplane arrangements (hence rediscovering Winder's result), and by M. Las Vergnas in acyclic reorientations of oriented matroids (result equivalent to a generalization of Winder's result to pseudohyperplane arrangements).

UPDATE

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october 2005

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ADVANCES IN Applied Mathematics
Advances in Applied Mathematics 32 (2004) 212–238
www.elsevier.com/locate/yaama

Bases, reorientations, and linear programming, in uniform and rank-3 oriented matroids

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Received 7 February 2002; accepted 10 March 2003

Abstract

A comparison of two expressions of the Tutte polynomial of an ordered oriented matroid, one as a generating function of basis activities, the other as a generating function of reorientation activities, yields a remarkable numerical relation between the number of bases and reorientations with given activities. The object of the paper is a natural activity preserving correspondence with suitable multiplicities between bases and reorientations, constituting a bijective proof of this relation. The general construction will be published elsewhere. In the present self-contained paper, we consider into details two particular cases of special interest: uniform oriented matroids and acyclic oriented matroids of rank 3. In both cases, the construction is simpler than in the general case, but introduces some of the main ideas. The correspondence is closely related to oriented matroid programming, a combinatorial generalization of linear programming. The link is direct in the uniform case: for unitary activities, the correspondence amounts to applying a program or its opposite to all bounded regions of a simple arrangement of pseudohyperplanes. In the rank-3 case, equivalent to pseudoline arrangements, a second step toward the general construction is made: optimizing two nested faces with respect to two lexicographically ordered programs.
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Keywords: Oriented matroid; Matroid; Tutte polynomial; Basis; Basis activity; Orientation; Acyclic orientation; Pseudoline; Pseudoline arrangement; Linear programming; Bijective proof

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doi:10.1016/S0196-8858(03)00081-2

1. Introduction

The *Tutte polynomial* of a matroid is a 2-variable polynomial invariant, introduced for graphs by W.T. Tutte in [16], and generalized to matroids by H.H. Crapo in [4]. Up to simple algebraic transformations, the Tutte polynomial of a matroid is equivalent to its *rank-generating function*, i.e., to the generating function of cardinality and rank of subsets of elements. The Tutte polynomial is a fundamental tool in the theory of numerical invariants of matroids, and has numerous applications. We refer the reader Section 2 for relevant definitions, and to [3] for an extensive survey on the subject.

Let M be a matroid on a linearly ordered set of elements E. By a theorem proved by W.T. Tutte for graphs [16], and extended to matroids by H.H. Crapo [4], we have

$$t(M; x, y) = \sum_{i,j} b_{i,j} x^i y^j$$

where $b_{i,j}$ is the number of bases of M such that i basis elements are smallest in their fundamental cocircuit and j nonbasis elements smallest in their fundamental circuit.

On the other hand, if M is an oriented matroid, M. Las Vergnas has shown in [13] that

$$t(M; x, y) = \sum_{i,j} o_{i,j} 2^{-i-j} x^i y^j$$

where $o_{i,j}$ is the number of reorientations of M such that i elements are smallest in some positive cocircuit and j elements smallest in some positive circuit. This last formula contains several results of the literature on counting acyclic (re)orientations in graphs, matroids, and regions in arrangements of (pseudo)hyperplanes [2,9–12,15,17] (see Section 2).

Comparing these two expressions for $t(M; x, y)$, we get the relation

$$o_{i,j} = 2^{i+j} b_{i,j}$$

for all i, j. A natural question arises of a bijective interpretation of this formula [13]. The problem is to define a correspondence between bases and reorientations, preserving parameters (i, j), called *activities*, and compatible with the above formula. More precisely, the desired correspondence should associate with a (i, j)-*active* basis of M, a set of 2^{i+j} (i, j)-active reorientations, in such a way that each reorientation of M is in the image of a unique basis.

We construct in the forthcoming paper [8] (see also [6]) a correspondence with these properties for general oriented matroids, the *canonical active correspondence*. In the present paper, we present into details two special cases, namely when the oriented matroid is uniform (Section 3) and when it is acyclic of rank 3 (Section 4). In these two cases, proofs are significantly simpler than in the general case, and particular properties occur, justifying a separate treatment. Another case with specific properties, the graphical case, is presented in [7].

The canonical active correspondence can be constructed in several different ways. A construction by decomposition of activities reduces the problem to the case of *unitary*—i.e., $(1,0)$ or $(0,1)$—activities. In this case the correspondence can be characterized intrinsically, or constructed by means of an algorithm. The general characterization simplifies in the uniform and rank-3 cases. We prove in both cases that the canonical active correspondence has the desired properties (Theorems 3.2, 3.8, 4.2, 4.6). As frequently in the context of Tutte polynomials, a deletion/contraction construction exists (Proposition 3.10 in the uniform case).

The canonical active correspondence is natural in several respects. In particular, its geometric interpretation in terms of the topological representation of oriented matroids establishes a close relationship with oriented matroid programming. Let M be a rank-r uniform oriented matroid on a linearly ordered set $E = \{e_1 < e_2 < \cdots\}$. We consider the topological representation of M by a simple arrangement of pseudohyperplanes with plane at infinity e_1. Let $A \subseteq E \setminus \{e_1\}$ be a $(1,0)$-reorientation of M. Then A being acyclic corresponds to a region R of the arrangement, and since its dual-orientation activity is 1 this region R is bounded. Suppose R is on the positive side of e_2. The matroid program on the bounded region R with plane at infinity e_1 and objective function e_2, nondegenerate since the arrangement is simple, has a unique solution at a vertex v of R. Then the canonical active correspondence associates with A the basis $B = \{e_1, b_2, \dots, b_r\}$, where $b_2, \dots, b_r$ are the $r-1$ pseudohyperplanes of the simple arrangement containing v. The hyperoctant with apex v containing R is uniquely determined among the 2^{r-1} hyperoctants defined by $b_2, \dots, b_r$ by the property of having a bounded intersection with e_2.

In the rank-3 case, the topological representation is an arrangement of pseudolines. The geometric interpretation in terms of oriented matroid programming is similar, but more involved for two reasons. First, the program may be degenerate, with an edge solution instead of a vertex solution. Using a second smallest objective function, we can still define uniquely the apex v of the region R. A second difficulty arises from the fact that we may have any number of pseudolines through v, hence the vertex v is not sufficient to determine R. An edge of the border of R containing v has to be determined, by optimization with respect to the linear ordering. We mention that for nonuniform oriented matroids of rank $\geqslant 4$, not considered here, a further difficulty occurs when v is a nonsimple vertex of R.

In view of the relation $o_{1,0} = 2b_{1,0}$, to prove bijectivity in the unitary case it suffices to prove either injectivity or surjectivity. In Sections 3 and 4, we prove both, thus providing a natural bijective proof of this formula. The case of general (i,j) activities is derived from the $(1,0)$ case by means of decompositions of activities for both matroid bases and oriented matroids. Decompositions of activities are outlined in the case of graph orientations in [14], appear partly for matroid bases in [5], and are described in [8] (see also [6]) in full generality. In the special cases of the present paper, general definitions can be avoided by means of direct constructions.

Finally, we mention that in the two particular cases of the paper the canonical active correspondence for $(1,0)$ activities is uniquely determined by the bijectivity property and an incidence preserving property (Propositions 3.10 and 4.7). This property does not hold in general.

2. Notation and terminology

Let M be a matroid on a set of elements E, and $B \subseteq E$ be a basis of M. For $e \in E \setminus B$, we denote by $C(B; e)$ the *fundamental circuit* of e with respect to B, i.e., the unique circuit contained in $B \cup \{e\}$. Dually, for $e \in B$, we denote by $C^*(B; e)$ the *fundamental cocircuit* of e with respect to B, i.e., the unique cocircuit contained in $(E \setminus B) \cup \{e\}$. For $e \in E \setminus B$ and $e' \in B$, we have clearly $e' \in C(B; e)$ if and only if $e \in C^*(B; e')$, and then $C(B; e) \cap C^*(B; e') = \{e, e'\}$.

We say that a matroid M is *ordered* if its set of elements E is linearly ordered. The notion of *activities* of a basis B in an ordered matroid M is due to W.T. Tutte [16] in the case of graphs, and to H.H. Crapo [4] in the case of matroids. The *internal activity* $\iota(B)$ is the number of elements $e \in B$ smallest in their fundamental cocircuit $C^*(B; e)$, and the *external activity* $\epsilon(B)$ is the number of elements $e \in E \setminus B$ smallest in their fundamental circuit $C(B; e)$. We say that a basis B with $\iota(B) = i$ and $\epsilon(B) = j$ is an (i, j)*-basis*. We denote by $b_{i,j}(M)$ the number of (i, j)-bases of M.

Spanning tree activities have been introduced by Tutte to generalize, in a self-dual way, classical properties of the chromatic polynomial of a graph [16]. The theorem for graphs extends to matroids [4], we have

$$t(M; x, y) = \sum_{i,j} b_{i,j} x^i y^j.$$

This expression readily implies that the coefficients $b_{i,j}$ are independent from the ordering of E. In recent textbooks, the Tutte polynomial of a matroid is defined by the closed formula

$$t(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(M)-r_M(A)} (y-1)^{|A|-r_M(A)}$$

algebraically equivalent to the *rank generating function* of the matroid, and the above formula is proved by deletion/contraction of the greatest element (see [3]).

For usual definitions on oriented matroids, the reader is referred to [1]. If the matroid M is oriented for $e \in E \setminus B$, we denote by $C(B; e)$ the unique signed circuit C contained in $B \cup \{e\}$ such that $e \in C^+$, and dually for $e \in B$, we denote by $C^*(B; e)$ the unique signed cocircuit D contained in $(E \setminus B) \cup \{e\}$ such that $e \in D^+$. We will sometimes, when it is not ambiguous, make the abuse of notation consiting of using the same letter for a signed circuit or cocircuit and its (unsigned) support.

An oriented matroid is *acyclic* if it contains no positive circuit, or equivalently, if every element is contained in a positive cocircuit. Dually, an oriented matroid is *totally cyclic* if it contains no positive cocircuit, or equivalently, if every element is contained in a positive circuit. An oriented matroid is acyclic if and only if the dual oriented matroid is totally cyclic.

A basic result in the domain of the present paper, is a theorem of R. Stanley [15]: the number of acyclic orientations of a graph G is equal to $t(C(G); 2, 0)$, where $C(G)$ is the cycle matroid of G [15]. This theorem has been generalized independently in 1975 by

T. Zaslavsky to real spaces in terms of arrangements of hyperplanes [17] (see also [2]), and by M. Las Vergnas to oriented matroids [10].

The paper [13] introduces a generalization of these results in terms of an *orientation generating function*. The (*primal*) *orientation activity* of an ordered oriented matroid M, or O*-activity*, denoted by $o(M)$, is the number of elements smallest in some positive circuit. The *dual orientation activity* of M, or O^**-activity*, denoted by $o^*(M)$, is the number of elements smallest in some positive cocircuit. We denote by $o_{i,j}(M)$ the number of subsets $A \subseteq E$ such that $o^*(-_A M) = i$ and $o(-_A M) = j$, where $-_A M$ denotes the *reorientation* of M obtained by reversing signs on A (this notation differs slightly from the notation $-_A M$ used in [1]). If no confusion results, for brievity, we sometimes say that the set A itself is a *reorientation* (we point out that different reorientations A may produce the same reoriented matroid $-_A M$), and that a reorientation A such that $o^*(-_A M) = i$ and $o(-_A M) = j$ is a (i, j)*-reorientation*. The definitions of O- and O^*-activities have been introduced in [13] in relation with the formula

$$t(M; x, y) = \sum_{i,j} o_{i,j} 2^{-i-j} x^i y^j.$$

This formula implies that $o_{i,j}$ does not depend on the ordering, and that $o_{i,j} = 2^{i+j} b_{i,j}$. The proof in [13] is by deletion/contraction of the greatest element. Note that $\sum_i o_{i,0}$ is the number of acyclic reorientations of M, hence the above formula generalizes results of [2,10,15,17].

The proofs of Theorems 3.4 and 4.2 below use the equality $o_{1,0} = 2b_{1,0}$, which is a particular case of the above result for the orientation generating function. This special case is originally due to C. Greene and T. Zaslavsky [9] for acyclic orientations of graphs with adjacent unique source and sink (see [7]), or bounded regions in real spaces, a result generalized in [11] to oriented matroids.

The paper uses extensively the topological representation of oriented matroids. Some knowledge of oriented matroid programming is also necessary. We refer the reader to [1, Chapters 5 and 10] for the needed prerequisites.

3. Uniform oriented matroids

We begin this section by stating the founding property of the general canonical active correspondence. It simplifies in the cases studied in this paper.

Proposition 3.0. *Let M be an oriented matroid on a linearly ordered set E, and B be a $(1, 0)$-active basis of M. Set $B = \{b_1 < b_2 < \cdots < b_r\}$ and $E \setminus B = \{c_1 < c_2 < \cdots < c_{n-r}\}$.*

Then there exist a unique pair of opposite reorientations A and $E \setminus A$ such that, setting $M' = -_A M = -_{E\setminus A} M$,

(i) *the covectors $C^*_{M'}(B; b_1)$, $C^*_{M'}(B; b_1) \circ C^*_{M'}(B; b_2)$, ..., $C^*_{M'}(B; b_1) \circ C^*_{M'}(B; b_2) \circ \cdots \circ C^*_{M'}(B; b_r)$ are positive, and*

(ii) *the vectors* $C_{M'}(B; c_1)$, $C_{M'}(B; c_1) \circ C_{M'}(B; c_2), \ldots, C_{M'}(B; c_1) \circ C_{M'}(B; c_2) \circ \cdots \circ C_{M'}(B; c_{n-r})$ *have the smallest element* b_1 *of* E *as unique negative element.*

Furthermore A *is a* $(1, 0)$*-reorientation of* M.

The canonical active basis-reorientation correspondence is defined on $(1, 0)$-bases of a general ordered oriented matroid M by associating with a $(1, 0)$-basis of M the two opposite $(1, 0)$-reorientations given by Proposition 3.0. The proof of Proposition 3.0 is less than one page long. Nevertheless, we omit it in the present paper, since Proposition 3.0 is quoted here only as a motivation (it will appear in [8], see also [6]). Applying Proposition 3.0 to the particular cases of uniform and acyclic rank-3 oriented matroids, we will derive simplified definitions for the canonical active correspondence, first from a combinatorial point of view, then in terms of the topological representation of oriented matroids and of oriented matroid programming, yielding short direct proofs of bijectivity (the general proof of bijectivity is about 4 page long). Of course, we could have given these definitions from scratch. We find it interesting to show how they are related, and proceed from the same general setting.

Two dual algorithms to construct a $(1, 0)$-reorientation A associated with a $(1, 0)$-basis B by the canonical active correspondence are easy corollaries of Proposition 3.0.

Algorithm 3.0.1. (1) *reorient in* $C^*_M(B; b_1)$ *to get all signs positive*;
(2) *for* $i = 2, \ldots, r$ *reorient in* $C^*_M(B; b_i) \setminus \bigcup_{j<i} C^*_M(B; b_j)$ *to get all signs opposite to the reoriented sign of the minimal element of* $C^*_M(B; b_i)$ (*this minimal element is necessarily in* $\bigcup_{j<i} C^*_M(B; b_j)$).

Algorithm 3.0.2. (1) *reorient in* $C_M(B; c_1)$ *to get* e_1 *negative and all other signs positive*;
(2) *for* $i = 2, \ldots, r$ *reorient in* $C_M(B; c_i) \setminus \bigcup_{j<i} C_M(B; c_j)$ *to get all signs opposite to the reoriented sign of the minimal element of* $C_M(B; c_i)$ (*this minimal element is necessarily in* $\bigcup_{j<i} C_M(B; c_j)$).

A rank-r matroid on n elements is *uniform* if its bases are all r-subsets of elements, or, equivalently, if its circuits are all $(r + 1)$-subsets of elements, or, equivalently, its cocircuits are all $(n - r + 1)$-subsets of elements. The abstract rank-r uniform matroid on n elements is denoted by $U_{r,n}$. Uniform nonoriented matroids are very simple objects, whereas uniform oriented matroids encompass a significant part of the general theory. In the present context, they provide a simple intuitive approach to the intricacies of the general case, specially from the linear programming point of view.

Let M be a uniform matroid on a linearly ordered set $E = \{e_1 < e_2 < \cdots\}$, and B be a $(1, 0)$-active basis. As easily seen, we have $\iota(B) = 1$ and $\epsilon(B) = 0$ if and only if $e_1 \in B$ and $e_2 \notin B$. Then a $(1, 0)$-basis B is determined by the fundamental cocircuit $D = C^*(B; e_1)$ of e_1: we have $B = (E \setminus D) \cup \{e_1\}$.

We apply Algorithm 3.0.1 to B. Since M is uniform, as sets we have $C^*(B; b_i) = (E \setminus B) \cup \{b_i\}$ and $C(B; c_j) = B \cup \{c_j\}$. In the first step of Algorithm 1, we reorient positively $D = C^*(B; b_1 = e_1)$ by reversing signs on D^-; note that $e_1 \notin D^-$. In step $i \geqslant 2$, we have reverse or not the sign of b_i if and only if b_i has the same sign that the reoriented e_2

in $C^*(B; b_i)$. If $e_2 \in D^+$ then the sign of e_2 is not changed, hence the sign of b_i is reversed if and only if e_2 is positive in the original cocircuit $C^*_M(B; b_i)$, hence by orthogonality if and only if b_i is negative in $C_M(B; e_2)$. The condition is reversed if $e_2 \in D^-$. Summing up, we get

Definition 3.1. Let M be a uniform oriented matroid on a linearly ordered set $E = \{e_1 < e_2 < \cdots\}$. We define the canonical active correspondence in the unitary case by associating with a $(1,0)$-active basis B the two opposite reorientations A and $E \setminus A$ defined by

$$A = (C^- \cup D^-) \setminus \{e_1\}$$

where $D = C^*(B; e_1)$ and $C = C(B; e_2)$ if $e_2 \in D^+$ respectively $C = -C(B; e_2)$ if $e_2 \in D^-$.

Note that in $-_A M$ the fundamental cocircuit D is positive and the fundamental (up to opposite) circuit C has $C^- = \{e_2\}$. We now establish that the reorientation is $(1,0)$-active and that the correspondence is bijective.

Theorem 3.2. *Let M be a uniform ordered oriented matroid. The canonical active correspondence is a bijection from the set of* $(1,0)$*-active bases of M to the set of pairs of opposite* $(1,0)$*-reorientations of M.*

Remark 3.2.1. (i) We have $-_A M = -_{E\setminus A} M$. Hence, the active basis-reorientation correspondence defines a bijection from the set of $(1,0)$-bases of M onto the set of reorientations M' of M with $(1,0)$ orientation activities.

(ii) The oriented matroid $-_A M$ depends only on the reorientation class of M. Applied to a reorientation M' of M the definition of Theorem 3.2 produces a set A' such that $-_{A'} M' = -_A M$.

(iii) The linear ordering on E is effective only by its first two elements $e_1 < e_2$. A permutation of $\{e_3, e_4, \ldots, e_n\}$ does not change the active correspondence on $(1,0)$-bases.

As well known, in an oriented matroid an element is either in a positive circuit, or in a positive cocircuit, but not in both. This property is sometimes called the Farkás Lemma for oriented matroids [1, Corollary 3.4.6].

Lemma 3.2.2. *Let M be a uniform oriented matroid on a linearly ordered set with smallest element e_1. The following properties are equivalent*:

(i) $o^*(M) = 1$;
(ii) *M contains a positive cocircuit, and a circuit C with $C^- = \{e_1\}$.*

Proof. We show that (i) implies (ii). If $o^*(M) > 0$ then by definition M contains a positive cocircuit. The condition $o^*(M) = 1$ means that all positive cocircuits contain e_1. It follows that M contains no cocircuit D with $D^- = \{e_1\}$, otherwise, by elimination, we get a

Table 1

	e	f	b	$B_1 \setminus B2$	$B_2 \setminus B_1$	$(B_1 \cap B_2) \setminus e$	$E \setminus (B_1 \cup B_2) \setminus f$
C_1	$-$	$+$	$+$	$+$	0	$+$	0
$-C_2$	$+$	$-$	0	0	$-$	$-$	0
C	$\pm/0$	0	$+$	$+/0$	$-/0$	$\pm/0$	0
D	0	$\pm/0$	$+$	$+/0$	$-/0$	0	$\pm/0$
$-D_1$	$-$	$-$	0	0	$-$	0	$-$
D_2	$+$	$+$	$+$	$+$	0	0	$+$

positive cocircuit not containing e_1. Hence by the Farkás Lemma for oriented matroids applied to $-_{e_1}M$, there is a circuit C with $C^- = \{e_1\}$.

Conversely, suppose M contains a circuit C with $C^- = \{e_1\}$, and let D be a positive cocircuit. We have $C \cap D \neq \emptyset$ since M is uniform. If $e_1 \notin D$ then all elements in $C \cap D$ are positive in C and in D, contradicting the orthogonality condition. □

Lemma 3.2.3. *In a uniform oriented matroid, for any fixed $e, f \in E$, there is at most one positive cocircuit D containing two elements e, f such that the circuit $C = (E \setminus D) \cup \{e, f\}$ has $C^- = \{e\}$.*

Proof. Suppose, by contradiction, there are two different bases B_1, B_2 containing e and not containing f such that the circuits $C_1 = B_1 \cup \{f\}$ and $C_2 = B_2 \cup \{f\}$ have $C_1^- = C_2^- = \{e\}$ and the cocircuits $D_1 = (E \setminus B_1) \cup \{e\}$ and $D_2 = (E \setminus B_2) \cup \{e\}$ are positive.

Let $b \in B_1 \setminus B_2 = C_1 \setminus C_2 = D_2 \setminus D_1$. Let C be a circuit obtained from C_1 and $-C_2$ by eliminating f, such that $b \in C$. We have $b \in C \subseteq (C_1 \cup C_2) \setminus \{f\} = B_1 \cup B_2$, $C \cap (B_1 \setminus B_2) \subseteq C^+$ and $C \cap (B_2 \setminus B_1) \subseteq C^-$. Let D be a cocircuit obtained from $-D_1$ and D_2 by eliminating e, such that $b \in D$. We have $b \in D \subseteq (D_1 \cup D_2) \setminus \{e\} = E \setminus (B_1 \cap B_2)$, $D \cap (B_1 \setminus B_2) \subseteq D^+$ and $D \cap (B_2 \setminus B_1) \subseteq D^-$ (see Table 1).

We have $b \in C \cap D \subseteq (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$. The signs of C and D coincide on their nonempty intersection, contradicting the orthogonality property. □

Proof of Theorem 3.2. If $e_2 \in D^-$ then set $C = -C(B; e_2)$, if $e_2 \in D^+$ then set $C = C(B; e_2)$. And set $D = C^*(B; e_1)$. Using orthogonality, since $C \cap D = \{e_1, e_2\}$, we have $e_1 \in C^- \cap D^+$, and the signs of e_2 in C and D are equal.

By definition, we have $A = (C^- \cup D^-) \setminus \{e_1\}$. It follows that $(-_A C)^- = \{e_1\}$ and $-_A D$ is positive. Hence by Lemma 3.2.2, we have $o^*(-_A M) = 1$. In a uniform oriented matroid, a circuit and a cocircuit have always a nonempty intersection, then using orthogonality, $-_A M$ has no positive circuit. Hence $-_A M$ is a $(1, 0)$-reorientation of M.

By Lemma 3.2.3, the mapping $B \mapsto A = (C^- \cup D^-) \setminus \{e_1\}$ is injective on the set of $(1, 0)$-bases of M. Hence this mapping is a bijection, since the number of $(1, 0)$-bases of M is equal to the number of subsets A of E such that $e_1 \notin A$ and $-_A M$ is a $(1, 0)$-reorientation of M [11]. □

We now give a topological interpretation of Theorem 3.2. We recall that, by the Topological Representation Theorem (see [1, Chapter 5]), the elements $\{e_1, e_2, \ldots, e_n\}$ of a rank-r

oriented matroid M can be represented by an arrangement of tame topological $(r-2)$-spheres, or *pseudospheres*, imbedded in $S = S^{r-1}$, with open halfspheres distinguished as e_i^+ and e_i^- for $i = 1, 2, \dots, n$, in such a way that the set of $\{0, +, -\}$-vectors defined by the signs of the pseudospheres on the vertices of the arrangement is identical to the set of cocircuits of M (see Example 3.2.1 below).

We denote by S^+ the closed halfsphere defined by e_1^+. We say that e_1 is the *infinity pseudosphere* or *plane at infinity* of S^+, and we restrict the pseudospheres $e_2, \dots, e_n$ to their intersections with S^+, called *pseudohyperplanes*. The *regions* of the arrangement are the connected components of the complement in S of the union of the pseudospheres $\{e_1, e_2, \dots, e_n\}$. A region is *bounded* if its closure does not meet e_1, or, equivalently, if none of its vertices belongs to e_1. The *sign-vector* of a region is the $\{+, -\}$-vector defined by the signs of the pseudospheres on any point of this region. The negative components of the sign-vectors define a bijection between the regions of the arrangement and the subsets $A \subseteq E$ such that $-_A M$ is an acyclic reorientation of M. In this bijection, the subsets A of E such that $e_1 \notin A$ and $-_A M$ is a $(1,0)$-reorientation of M, i.e., acyclic reorientations such that every positive cocircuit contains e_1, are in 1–1 correspondence with bounded regions contained in S^+. The number of bounded regions contained in S^+ is $b_{1,0}$ [9,11].

A $(1,0)$-basis B of M has the form $\{b_1 = e_1 < b_2 < \dots < b_r\}$, with $e_2 < b_2$. The pseudohyperplanes $b_2, \dots, b_r\}$ meet in a vertex v of the arrangement. The sign-vector of v is given by the fundamental cocircuit $D = C^*(B; e_1)$. Its $\pm$ signs constitute the sign-vector of the region containing v in the sub-arrangement constituted by the pseudohyperplanes not containing v. Since M is uniform, the sub-arrangement constituted by $b_1 = e_1, b_2, \dots, b_r$ and e_2 has a unique circuit $\{b_1 = e_1, e_2, b_2, \dots, b_r\}$, hence is homeomorphic to a real arrangement. Thus, we may suppose that b_i, $i = 2, \dots, r$, is homeomorphic to the coordinate hyperplane $x_{i-1} = 0$ of R^{r-1}, e_2 to the hyperplane $x_1 + x_2 + \dots + x_{r-1} = 1$, and e_1 to the plane at infinity. Using this homeomorphism, clearly, $b_2, \dots, b_r$ divide S^+ into 2^{r-1} *hyperoctants* with *apex* v, and exactly one of these hyperoctants, called the *active hyperoctant*, contains the unique bounded region determined by e_2 and $b_2, \dots, b_r$.

The fundamental cocircuit of $b_i \in B$ with respect to B correspond geometrically to the vertex intersection of $B \setminus b_i$. Set $C = \pm C(B; e_2)$ such that the sign of e_2 in C is the same than its sign in D. Namely, we have $C = C(B; e_2)$ if v is in e_2^+ and $C = -C(B; e_2)$ if $v \in e_2^-$. For $b_i \in B$, using orthogonality, the sign of b_i in the fundamental circuit of e_2 is the opposite of the sign of e_2 in the fundamental cocircuit of b_i. Hence the sign-vector of the active hyperoctant in the sub-arrangement constituted by the pseudohyperplanes containing v, is given by the signs in $C \setminus \{e_1, e_2\}$. Note that $e_1 \in C^- \cap D^+$. Summing up, the sign-vector of the unique region incident to v and contained in the active hyperoctant is given by the signs in $C \setminus \{e_1\}$ and D.

By Theorem 3.2, the active basis-reorientation correspondence associates with B the region R defined by the reorientation $A = (C^- \cup D^-) \setminus \{e_1\}$. Hence, we have proved

Proposition 3.3. *The region R of S^+ associated with a $(1,0)$-basis B of a uniform ordered oriented matroid by the active basis-reorientation correspondence is the unique region contained in the active hyperoctant defined by B and incident to its apex.*

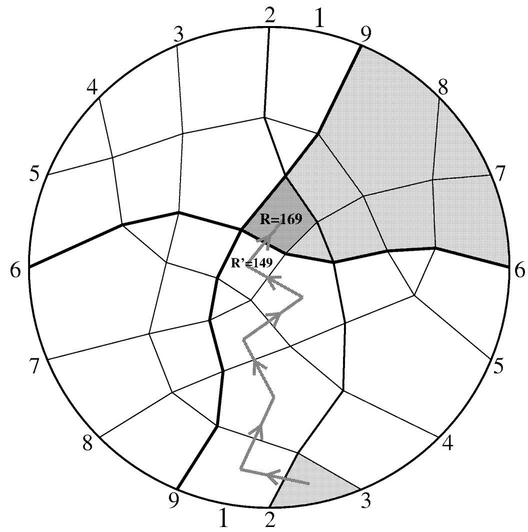


Fig. 1.

If the $\pm$ sides are defined by a *fundamental region*, positive in all pseudohyperplanes, then $A = (C^- \cup D^-) \setminus \{e_1\}$ is the set of pseudohyperplanes which have to be crossed to reach the region R from the fundamental region. More precisely, D^- permits to reach a region R' incident to v, and $C^- \setminus \{e_1\} \setminus D^-$ permits to go from R' to R. It follows from properties of oriented matroids [1], that these crossings can be rearranged in a path from the fundamental region to R', then to R (see below Example 3.3.1).

Example 3.3.1. The pseudoline arrangement of Fig. 1 is Ringel arrangement, a simple arrangement of 9 pseudolines derived from a non-Pappus configuration. We recall that Ringel arrangement is a *nonstretchable* arrangement (i.e., not combinatorially equivalent to an arrangement of lines) with the smallest possible number of pseudolines. The corresponding oriented matroid is uniform of rank 3 on 9 elements.

Signs are defined by a fundamental region of the arrangement (colored in lightgray, bottom of Fig. 1). We recall that the sign of an element x in a cocircuit $D = E \setminus \{e, f\}$ is $+$ if the fundamental region and the intersection of the pseudolines e and f are not separated by the pseudoline x, and $-$ if they are separated.

Let $B = 169$. The region R image of B by the active correspondence is colored in dark gray.

We read on Fig. 1 that $D = C^*(169; 1) = 1\bar{2}\bar{3}\bar{4}5\bar{7}\bar{8}$.

Signs of the circuit $C(169; 2)$ are defined by orthogonality, from the cocircuits meeting it in 1 and another element. We have already $1\bar{2}\bar{3}\bar{4}5\bar{7}\bar{8}$ with intersection 12. We read on Fig. 1 the cocircuits $1\bar{3}\bar{4}5\bar{6}\bar{7}\bar{8}$ for 16 and $1\bar{3}\bar{4}57\bar{8}9$ for 19. Therefore $C(169; 2) = 126\bar{9}$. It follows that $C = -C(169; 2) = \bar{1}\bar{2}\bar{6}9$, since $2 \in D^-$.

By Theorem 3.2 we have $A = (C^- \cup D^-) \setminus 1 = 234678$.

As easily seen on Fig. 1, the path 238476 goes from the fundamental region to $R' = 149$, then to $R = 169$ (there are other possible paths). In accordance with Proposition 3.3, the region R is the unique region contained in the active quadrant determined by the pseudolines 6 and 9, colored in mid gray in Fig. 1, and incident to their intersection.

Remark 3.3.2. Another way to define geometrically the region R associated with the given basis B is as follows. In Theorem 3.2, the reorientation A defining R is chosen so that in $-_A M$ the cocircuit $C^*(B; e_1)$ is positive, and e_1 is the only negative element in $C = \pm C(B; e_2)$. By orthogonality, e_2 and b_i have opposite signs in $C^*(B; b_i)$ for $i = 2, 3, \ldots, r$. Geometrically, this means, first, that the vertex v defined by $C^*(B; e_1)$ is incident to R. Then, the pseudo-simplex P determined by the pseudohyperplanes in B and contained in the positive side of e_2 is identical to the hyperoctant opposite to the active hyperoctant relatively to v. The region R being the region incident to v and opposite to P is the region incident to v contained in the active hyperoctant.

For an ordered uniform oriented matroid M on $E = \{e_1 < e_2 < \cdots\}$, the active basis-reorientation correspondence can be interpreted as a solution of an oriented matroid program (M, e_1, e_2) (see [1, Chapter 10] for oriented matroid programming) on each bounded region of the topological representation of M.

Proposition 3.4. *With above notation, the vertex v is the unique solution of the following oriented matroid program: maximize the objective function defined by e_2 if R is on the positive side of e_2, or minimize if R is on the negative side of e_2, on the bounded region R with respect to the infinity e_1.*

The definition in Theorem 3.2 is in disguise the 'simplex criterion' of [1, Corollary 10.2.8]. It follows that Proposition 3.4 is a reformulation of results of oriented matroid programming. For completeness, we give a direct proof in the present context.

The 'main theorem of oriented matroid programming' [1, Theorem 10.1.13] states that the graph of the program on a bounded region has at least one sink, unique in the nondegenerate case. We recall that given a plane at infinity e_1 and an objective function e_2 the *graph of the program* on a bounded region R is the partially directed graph defined by the vertices and edges of R such that an edge joining two adjacent vertices is directed in the increasing direction of the objective function [1, Definition 10.1.16].

We introduce a closely related graph, more convenient for our purpose.

Definition 3.5. The *active cocircuit graph G* of an ordered oriented matroid M is a directed graph whose vertex-set is the set of (signed) cocircuits of M. Two vertices D_1 D_2 are adjacent in G if and only if $E \setminus D_1$ and $E \setminus D_2$ are comodular in M[2] and D_1 and D_2 are conformal signed sets.[3]

[2] Two subsets of elements X_1 X_2 are *comodular* (short for *constitute a modular pair*) in a matroid M if and only if $r_M(X_1) + r_M(X_2) = r_M(X_1 \cap X_2) + r_M(X_1 \cup X_2)$. The complement $E \setminus D$ of a cocircuit D is an hyperplane, i.e., a flat of rank $r - 1$, of M, and conversely. Two different hyperplanes H_1 and H_2 are comodular in M if and only if the rank of $H_1 \cap H_2$ is $r - 2$, i.e., if and only if $H_1 \cap H_2$ is a coline.

[3] Two signed sets are *conformal* if and only if their signs coincide on their intersection.

The simplest definition of edge directions in G is in terms of the topological representation of M. Let D_1 D_2 be two cocircuits adjacent in G. Since $E \setminus D_1$ and $E \setminus D_2$ are comodular in M, then $L = E \setminus (D_1 \cup D_2)$ is a coline of M, i.e., a corank 2 flat. By properties of the pseudosphere arrangement representing M, the intersection of the pseudospheres in L is a pseudocircle $\lambda \approx S^1$, such that the intersections of λ with the pseudospheres in $E \setminus L = D_1 \cup D_2$ constitute an arrangement of 0-spheres, i.e., pairs of points, representing the rank 2 oriented matroid M/L. Let $\{e < e'\}$ be the lexicographically minimal basis of M/L. The two 0-spheres representing e and e' in λ divide λ into 4 topological segments, each with one extremity belonging to the 0-sphere e and the other to the 0-sphere e'. We direct these 4 segments from e' towards e. The conformal cocircuits D_1 and D_2 are combinatorially consecutive points of λ, i.e., each belongs to a 0-sphere, and the interior of one of the two topological segments they define, say δ, meets no other 0-sphere of the arrangement. Therefore, δ is contained in exactly one of the four segments defined by e and e', say σ. We direct the edge $D_1 - D_2$ in the direction of δ consistent with the direction of σ.

Example 3.3.1 (*continued*). Definition 3.5 is illustrated in rank 3 by Fig. 2. In this rank-3 example, since $3 - 2 = 1$, the pseudolines (and circle e_1) are both the pseudospheres representing the elements of the matroid and the pseudocircles of Definition 3.5. The edges of G are realized as topological segments of the pseudolines or pseudocircle.

For edges $D_1 - D_2$ of G with D_1 D_2 not both in e_1 or both in e_2, we have $e = e_1$ and $e' = e_2$ in Definition 3.5. For edges $D_1 - D_2$ supported by e_1, we have $e = e_2$ and $e' = e_3$. For edges $D_1 - D_2$ supported by e_2, we have $e = e_1$ and $e' = e_3$.

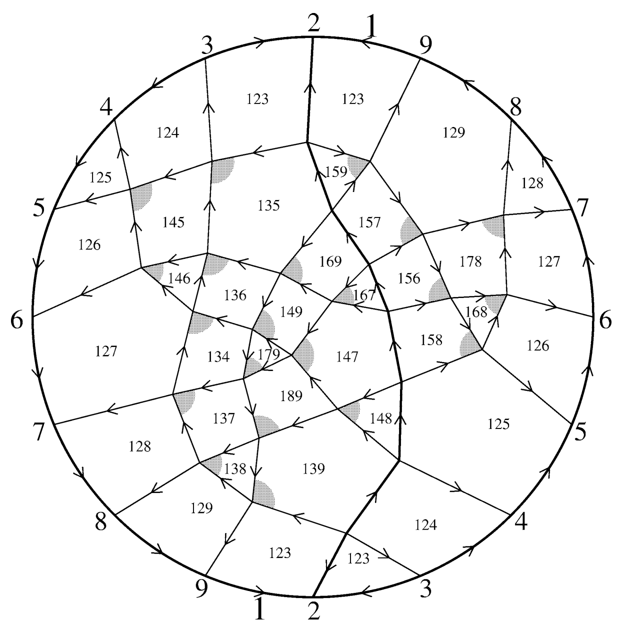


Fig. 2.

From Definition 3.5, we easily get a combinatorial definition, to be used in the proof of Proposition 3.4, of directions of edges of bounded regions in the particular case of a uniform oriented matroid. In this case, we have $e = e_1$ and $e' = e_2$.

Definition 3.5.2. Let $D_1 - D_2$ be an edge of the active cocircuit graph such that $e_1 \in D_1 \cap D_2$. Since M is uniform, we have $|D_1 \setminus D_2| = |D_2 \setminus D_1| = 1$, say $D_1 \setminus D_2 = \{x_1\}$ and $D_2 \setminus D_1 = \{x_2\}$. Then, *we direct the edge from D_1 to D_2 if*

- *$e_2 \notin D_1$ and $e_2 \in D_2$,*
- *or, $e_2 \in D_1 \cap D_2$, and we have $D(x_1) = D_1(x_1)$,[4] or, equivalently, $D(x_2) = -D_2(x_2)$, where D is the unique cocircuit obtained from D_1 D_2 by eliminating e_1, such that $D(e_2) = D_1(e_2) = D_2(e_2)$.*

In terms of Definition 3.5, the cocircuit D is the extremity of the segment σ which belongs to the 0-sphere e.

The active cocircuit graph coincide with the graph of a program on bounded regions located on the positive side of e_2, and has opposite edge directions on bounded regions located on the negative side of e_2. In the active cocircuit graph, no distinction is made between a minimum (a source in the program graph) and a maximum (a sink in the program graph). This slight change has an important consequence in our context. In the general case, several simultaneous linear programs have to be handled, with a mixture of minimizing and maximizing [8] (see also [6]). For instance, in rank 3 (see Section 4), we have to consider two matroid linear programs in the degenerate cases (with respect to e_2 and e_m). The main point is that vertices produced by the active basis-reorientation correspondence are always associated with sinks of the active cocircuit graph, whereas this would not be the case for program graphs, or their natural extensions to the whole set of cocircuits. We point out that the active cocircuit graph depends on the ordering, but is invariant under reorientation.

Proof of Proposition 3.4. Let R be a bounded region. Since the active cocircuit graph G is invariant under reorientation, without loss of generality we may suppose that R is the fundamental region of the arrangement. Let v_0 be a vertex of R, unique by Lemma 3.2.2, such that the corresponding cocircuit D_0 is positive, and the circuit $C_0 = (E \setminus D_0) \cup \{e_1, e_2\}$ has $C_0^- = \{e_2\}$. With Theorem 3.2, we know that there exist such a vertex: it corresponds to the cocircuit D for the $(1,0)$-basis associated with R.

Suppose there is an edge $D_0 \to D_2$ in the graph G such that D_2 is a vertex of R, i.e., is a positive cocircuit. Set $D_0 \setminus D_2 = \{x_0\}$ and $D_2 \setminus D_0 = \{x_2\}$. Let D be the unique cocircuit contained in $E \setminus D_0 \cup D_2$ such that $e_1 \notin D$ and $e_2 \in D^+$. By Definition 3.5.2, we have $x_2 \in D^-$. It follows that C_0 and D have opposite signs on their intersection $C_0 \cap D = \{e_2, x_2\}$, contradicting the orthogonality property. (See Table 2.)

Let D_1 be a positive cocircuit different from D_0. We show that in G the vertex D_1 has at least one outgoing edge. We have $e_1 \in D_1$ since R is bounded. If $e_2 \notin D_1$, then

[4] Let X be a signed set, and $e \in X$. Then $X(e)$ denotes the sign of e in X. We have $X(e) = 1$ if and only if $e \in X^+$, $X(e) = -1$ if and only if $e \in X^-$.

Table 2

	e_1	e_2	x_0	x_2	$D_0 \setminus \{e_1, e_2, x_0\}$	$E \setminus (D_0 \cup D_2)$
D_0	$+$	$+$	$+$	0	$+$	0
D_2	$+$	$+$	0	$+$	$+$	0
D	0	$+$	$+$	$-$	$\pm/0$	0
C_0	$+$	$-$	0	$+$	0	$+$

Table 3

	e_1	e_2	x	$D_1 \setminus \{e_1, e_2\}$	$E \setminus D_1 \setminus \{x\}$
C_1	$-$	$+$	$-$	0	$\pm/0$
D_1	$+$	$+$	0	$+$	0
D	0	$+$	$+$	$\pm/0$	0
D_2	$+$	$+$	$+$	$+/0$	0

for any positive cocircuit D_2 with $e_2 \in D_2$ comodular with D_1, we have $D_1 \to D_2$ by the first case of Definition 3.5.2. Suppose $e_2 \in D_1$. Let C_1 be the circuit supported by $(E \setminus D_1) \cup \{e_1, e_2\}$ such that $e_2 \in C_1^+$. We have $C_1 \cup D_1 = \{e_1, e_2\}$, hence $e_1 \in C_1^-$. By Lemma 3.3.2, there is $x \in C_1 \setminus \{e_1, e_2\}$ such that $x \in C_1^-$. Let D be the cocircuit supported by $D_1 \setminus \{e_1\} \cup \{x\}$ such that $x \in D^+$. Since $x \in C_1 \cap D \subseteq \{e_2, x\}$, by orthogonality we have $C_1 \cap D = \{e_2, x\}$, hence $e_2 \in D^+$. The composition $D_1 \circ D$ is a positive covector. Hence, by conformal composition,[5] there is a positive cocircuit D_2 such that $x \in D_2$. We have $e_1 \in D_2$ since R is bounded. Since $\{e_1, x\} \subseteq C_1^- \cap D_2^+\}$ and $C_1 \cap D_2\} \subseteq \{e_1, e_2, x\}$, by orthogonality, we have $e_2 \in D_2 = D_2^+$. Therefore, by the second case of Definition 3.5.2, we $D_1 \to D_2$. (See Table 3.) □

We point out that Theorem 3.3 and Proposition 3.4 provide as corollary an alternate proof of the main theorem of oriented matroid programming in the nondegenerate case. Conversely Proposition 3.4 and the main theorem of oriented matroid programming show that the active correspondence is surjective.

We mention that the duality between circuits and cocircuits in Theorem 3.2 is related to duality in linear and oriented matroid programming (see [1, Proposition 10.1.4]).

We now extend the active correspondence from the $(1, 0)$ case to the general case. The main tool is a partition of set of elements of the oriented matroid, called *active partition*, either with respect to a basis in an ordered matroid or with respect to the orientation in an ordered oriented matroid. Active partitions permit to reduce general (i, j) activities to $(1, 0)$ (or, dually, $(0, 1)$) activities, by means of associated minors, and to extend consistently the canonical active correspondence from $(1, 0)$-active bases to all bases [8] (see also [6]). In the uniform case, active partitions and the corresponding construction can be described directly very easily.

[5] The *composition* $X \circ Y$ of two signed sets X, Y is defined by $(X \circ Y)^+ = X^+ \cup (Y^+ \setminus X)$ and $(X \circ Y)^- = X^- \cup (Y^- \setminus X)$. In an oriented matroid, any composition of circuits respectively cocircuits, is a conformal composition of circuits respectively cocircuits [1, Proposition 3.7.2].

Proposition 3.6. *Let $E = \{e_1 < e_2 < \cdots < e_n\}$ be a linearly ordered set, and $M \approx U_{r,n}$ be a rank-r uniform matroid on E.*

(i) *A basis B is either internal—if $e_1 \in B$, or external—if $e_1 \notin B$.*
(ii) *If $e_1 \in B$, and $r < n$, let i be the smallest integer such that $e_{i+1} \notin B$, then $\iota(B) = i$, and the internally active elements of B are $e_1, e_2, \ldots, e_i$, there is no externally active elements. The basis $B \setminus \{e_1, \ldots, e_{i-1}\}$ of $M/\{e_1, \ldots, e_{i-1}\}$ is $(1, 0)$-active.*

The proof is easy and left to the reader. In the case of (ii), we call *active partition* with respect to B the partition $E = \{e_1\} + \{e_2\} + \cdots + \{e_{i-1}\} + (E \setminus \{e_1, e_2, \ldots, e_{i-1}\})$.

It follows that for $0 < r < n$, we have

$$b_{i,0}(U_{r,n}) = \sum_{i=1}^{i=r} \binom{n-i-1}{r-i}, \qquad b_{0,j}(U_{r,n}) = \sum_{j=1}^{j=n-r} \binom{n-j-1}{n-r-j},$$

and $b_{i,j}(U_{r,n}) = 0$ for $i, j > 0$.

Hence, for $0 < r < n$,

$$t(U_{r,n}; x, y) = \sum_{i=1}^{i=r} \binom{n-i-1}{r-i} x^i + \sum_{j=1}^{j=n-r} \binom{n-j-1}{n-r-j} y^j.$$

Special cases: $t(U_{n,n}; x, y) = x^n$ and $t(U_{0,n}; x, y) = y^n$.

Proposition 3.7. *Let M be an ordered uniform oriented matroid on a linearly ordered set $E = \{e_1 < e_2 < \cdots < e_n\}$.*

(i) *M is either acyclic or totally cyclic.*
(ii) *Suppose M acyclic with $o^*(M) = i$. Then the O^*-active elements of M are $e_1, e_2, \ldots, e_i$, and $M/\{e_1, e_2, \ldots, e_i\}$ is $(1, 0)$-active.*

The orientation active partition of M is $E = \{e_1\} + \{e_2\} + \cdots + \{e_{i-1}\} + \{e_i, e_{i+1}, \ldots, e_n\}$.

Proof. (i) This elementary property is well known. We give a proof for completeness. Suppose M contains a positive cocircuit D, and let V be any positive covector containing D. Suppose $E \setminus V \neq \emptyset$, and let $e \in E \setminus V$. The matroid M being uniform, there is a cocircuit D' such that $D' \setminus V = \{e\}$ with $e \in D'^+$. Then $V' = V \circ D'$ is a positive vector with $|V'| = |V| + 1$. It follows inductively that E is a positive covector of M, i.e., M is acyclic.

(ii) It suffices to show that if e_j is O^*-active in M then e_{j-1} is also O^*-active. Suppose there is a positive cocircuit D with smallest element e_j. The matroid M being uniform, $D' = D \setminus \{e_j\} \cup \{e_{j-1}\}$ is also a cocircuit. Replacing if necessary D' by $-D'$, we may suppose $e \in D'^+$. Then $D \circ D'$ is a positive vector of M, hence by conformal composition a union of positive cocircuits of M. It follows that $e_{j-1} \in D \circ D'$ is in a positive cocircuit contained in $D \cup D'$, hence necessarily the smallest element of this cocircuit, and therefore is O^*-active. □

In view of Theorem 3.2, Propositions 3.6 and 3.7, the following theorem follows.

Theorem 3.8. *Let M be a uniform oriented matroid on linearly ordered set $E = \{e_1 < e_2 < \cdots < e_n\}$, and B be a basis of M.*

If $\iota(B) = i \geqslant 1$ (hence $\epsilon(B) = 0$), then the canonical active correspondence associates with B the 2^i $(i, 0)$-active reorientations $A \subseteq E$ of the form $A = X \cup A'$ and $A = X \cup (E \setminus A')$, such that $X \subseteq \{e_1, e_2, \ldots, e_{i-1}\}$ and A' is associated with the $(1, 0)$-basis $B \setminus \{e_1, e_2, \ldots, e_{i-1}\}$ in $M/\{e_1, e_2, \ldots, e_{i-1}\}$ by the canonical active correspondence.

If $\epsilon(B) = i \geqslant 1$ (hence $\iota(B) = 0$), then the canonical active correspondence associates with B the 2^i $(0, i)$-active reorientations $A \subseteq E$ associated with the $(i, 0)$-active basis $E \setminus B$ in M^*.

Then, each of the 2^n reorientations of M is associated with exactly one basis of M.

Remarks 3.8.1. (i) We point out that the canonical active correspondence not only preserves activities, which was our initial requirement, but also preserves active elements, and in fact preserves active partitions.

(ii) In an oriented matroid M with $o^*(M) = i$ and $o(M) = j$, we define an *activity class of reorientations* as the set of 2^{i+j} reorientations obtained by reversing signs on arbitrary unions of parts of the orientation active partition of M. The activity classes of reorientations obviously partition the set of 2^n reorientations of M. In the previous definition, as in the general case, the reorientations associated with a basis constitute an activity class of reorientations. The canonical active correspondence can be seen as an activity preserving bijection between bases and activity classes of reorientations.

(iii) As in Theorem 3.2, the ordering is effective only for the first elements. Changing the ordering of elements e_i with $i > \mathrm{Max}(r, n - r)$ does not modify the correspondence.

(iv) Propositions 3.4 and 3.7 provide the reverse correspondence.

Example 3.3.1 (*continued*). In Fig. 2, the basis associated with a region is indicated within the region. A dashed angle indicates the vertex, solution of the linear program on a bounded region. In a bounded region associated with a basis $\{e_1, b_2, b_3\}$, the two pseudolines supporting the angle of the region are b_2 and b_3.

We conclude this section by two properties of the active basis-reorientation correspondence. The first one provides an inductive construction of this correspondence. The second one exhibits natural properties determining uniquely the active basis-reorientation correspondence for realizable uniform oriented matroids.

We have shown that constructing the active basis-reorientation correspondence on bounded regions, i.e., $(1, 0)$ acyclic oriented matroid M, is equivalent to constructing the sink of the active cocircuit graph on each bounded region, or, equivalently, the fundamental cocircuit of e_1 with respect to the basis associated with M. For short, we denote this fundamental cocircuit by $\mathrm{Opt}(M)$.

Proposition 3.9. *Let M be a $(1,0)$ orientation active ordered uniform oriented matroid on $E = \{e_1 < e_2 < \cdots\}$. Let R be a bounded region representing M in a topological representation by a pseudosphere arrangement, let $e \in E \setminus \{e_1, e_2\}$, and $-_e R$ denote the region obtained by crossing the pseudosphere e from R if $-_e M$ is acyclic.*

The application Opt *is uniquely determined by the following induction*:

(i) *If $|E| = 2$, then R is reduced to the optimal vertex, then* $\mathrm{Opt}(M) = \{e_1, e_2\}$.
(ii) *If $-_e M$ is not acyclic, then* $\mathrm{Opt}(M) := \mathrm{Opt}(M \setminus e) \cup \{e\}$.
(iii) *If $-_e R$ is an unbounded region, then* $\mathrm{Opt}(M) := \mathrm{Opt}(M/e)$.
(iv) *If $-_e R$ is a bounded region, there are two cases*:
- *the optimal vertex of the region containing R in the arrangement obtained by deleting e is incident to R, then* $\mathrm{Opt}(M) := \mathrm{Opt}(M \setminus e) \cup \{e\}$;
- *the optimal vertex of R is on the pseudosphere e, then* $\mathrm{Opt}(M) := \mathrm{Opt}(M/e)$.

Proof. The proof is by induction on $|E|$.

(i) The proposition is obvious when $|E| = 2$. Suppose $|E| \geqslant 3$.

Since the fundamental cocircuits of $B \setminus \{e\}$ in M/e if $e \in B$, respectively circuits of $M \setminus e$ if $e \notin B$, are the fundamental cocircuits, respectively circuits, of B in M with e removed, it follows immediately from the definition that if $e \in \mathrm{Opt}(M)$ then $\mathrm{Opt}(M) = \mathrm{Opt}(M/e)$, and if $e \notin \mathrm{Opt}(M)$ then $\mathrm{Opt}(M) = \mathrm{Opt}(M \setminus e) \cup \{e\}$.

(ii) If $-_e M$ is not acyclic then e belongs to every positive cocircuit of M.

By the definition of the active cocircuit graph, if D and D' are comodular positive cocircuits of M and e is both in D and D' then the edge $D - D'$ is directed from D to D' if and only if it is directed from $D \setminus \{e\}$ to $D' \setminus \{e\}$ in $M \setminus e$. So the active cocircuit graph M restricted to positive cocircuits of M is the same as in $M \setminus e$.

Then by Proposition 3.4 and Definition 3.5 $\mathrm{Opt}(M) = \mathrm{Opt}(M \setminus e) \cup \{e\}$.

(iv) Since there is a unique optimal vertex $\mathrm{Opt}(M)$ for any $(1,0)$-uniform oriented matroid, it follows from our preliminary observation and the induction hypothesis, that we have $\{\mathrm{Opt}(M), \mathrm{Opt}(-_e M)\} = \{\mathrm{Opt}(M/e), \mathrm{Opt}(M \setminus e) \cup \{e\}\}$.

Hence, if $\mathrm{Opt}(M \setminus e) \cup \{e\}$ is a positive cocircuit of M and we have $\mathrm{Opt}(M)$, otherwise $\mathrm{Opt}(-_e M)$ is a positive cocircuit and we have $\mathrm{Opt}(M) = \mathrm{Opt}(M/e)$.

(iii) A bounded region in $M \setminus e$ either is a bounded region in M case (ii), or contains a bounded region in M and its opposite region with respect to e case (iv).

Hence by the induction hypothesis the $b_{1,0}(M \setminus e)$ cocircuits of M containing e_1, e_2, and e have been associated with regions in cases (ii) and (iv). So the remaining cocircuits, which are optimal for a region R such that $-_e R$ is unbounded, must contain e, that is must satisfy $\mathrm{Opt}(M) = \mathrm{Opt}(M/e)$. □

The algorithm of Proposition 3.9 is a set-theoretical extension of the numerical deletion/contraction relation $t(M; 1, 0) = t(M \setminus e; 1, 0) + t(M/e; 1, 0)$. Its proof is based on well-known geometrical observations from linear programming: the suppression of a variable e corresponds to the contraction of an element e, and the suppression of a constraint e corresponds to the deletion of an element e. Here this linear programming technique is applied simultaneously to all bounded regions.

This deletion/contraction procedure can be generalized to any oriented matroid [8] (see also [6]). It provides an alternate construction of the canonical active basis-reorientation correspondence, based on comparisons of activities and adjacency properties in place of optimization properties and active partitions.

We say that a mapping from the vertices, or, equivalently, signed cocircuits, of an oriented matroid to the regions of its topological pseudosphere representation is *incidence preserving* if a vertex is always incident to its image region. Let V be the set of vertices of an ordered oriented matroid M not contained in the pseudospheres e_1 or e_2. If M is uniform, the active basis-reorientation correspondence induces an incidence preserving bijection from the set V onto the set of bounded regions: a cocircuit D such that $e_1, e_2 \in D$ with $e_1 \in D^+$ is mapped to the bounded region in e_1^+ associated with the $(1,0)$-basis $B = D \cup \{e_1\}$.

Proposition 3.10. *Let M be an ordered uniform oriented matroid on $E = \{e_1 < e_2 < \cdots\}$. If the active cocircuit graph contains no directed cycle in the set V of cocircuits containing both e_1 and e_2, then there exists a unique incidence preserving bijection from V onto the set of bounded regions. Otherwise, there are at least two such bijections.*

Proof. Let f be an incidence preserving bijection from V onto the set of bounded regions.

Suppose the active cocircuit graph is acyclic on V. Then, it induces an ordering on V. The bijection f induces a mapping g from V into itself: we map $v \in V$ to the unique sink $g(v)$ of the bounded region $f(v)$. The matroid M being uniform, a vertex is a sink in at most one bounded region. Hence g is a bijection from V onto itself. Since f preserves incidences, by properties of oriented matroid programming [1, Chapter 10], the bijection g is augmenting: we have $v \leqslant g(v)$ for all $v \in V$. Plainly, there is unique augmenting bijection in a finite ordered set, namely the identity. It follows that g is the identity, hence f is unique.

Suppose now that there is a directed cycle $v_0 \to v_1 \to \cdots \to v_\ell = v_0$ of the active cocircuit graph with $v_0, v_1, \ldots, v_\ell \in V$. Let R_i be the unique bounded region with (unique) sink v_i for $i = 1, 2, \ldots, \ell$. Then, since M is uniform, the vertex v_{i-1} is also incident to R_i for $i = 1, 2, \ldots, \ell$. Hence the mapping f' defined by f for $v \in V \setminus \{v_0, v_1, \ldots, v_\ell\}$ and $f'(v_{i-1}) = R_i$ for $i = 1, 2, \ldots, \ell$ is a second incidence preserving bijection from V onto the set of bounded regions. □

The active cocircuit graph is in particular acyclic when the uniform oriented matroid is *realizable*, i.e., arises from a configuration of points in real space. In general uniform oriented matroids the active cocircuit graph may contain directed cycles. In fact, one important difficulty in oriented matroid programming, as compared to real linear programming, is that the graph of a program may contain directed cycles. The smallest example is the oriented matroid *EFM*(8), uniform of rank 4 on 8 elements [1, Example 10.4.1]. An oriented matroid program (M, e_1, e_2) on an acyclic oriented matroid M with infinity plane e_1 and objective function e_2 is said *Euclidean* if the graph of the program contains no directed cycle [1, Theorem 10.5.5], and *non-Euclidean* otherwise.

4. Acyclic oriented matroids of rank 3

By the Topological Representation Theorem for oriented matroids, the acyclic reorientations of a rank-3 oriented matroid are represented by the regions of an arrangement of pseudolines in the plane. Our purpose in this section is to describe geometrically the canonical active basis-reorientation correspondence for acyclic ordered oriented matroids of rank 3 in terms of pseudoline arrangements. For $(1,0)$-bases we derive from the combinatorial constructions given by Proposition 3.0 and its corollaries a geometric construction of the corresponding region. Then we give a simple direct proof of the bijectivity property. For general internal bases, the correspondence is obtained from certain minors. Up to parallel elements, these matroids are uniform of rank $\leqslant 2$, hence it suffices to apply results of Section 3 in very simple cases.

The constructions of this section constitute a first approach of the degenerate cases, and of the flag programming introduced in the general case [8] (see also [6]). In terms of optimization, in the rank-3 acyclic case, the basis associated with a bounded region is optimal for an extended linear program with respect to the total order. A second objective function is introduced to define the optimal vertex when the first one insufficient in certain degenerate cases. The optimal basis $\{e_1 < e_p < e_q\}$ a basis defines two nested faces $e_p \cap e_q$ and e_q which have to be optimized. Intuitively, the canonical active correspondence can be thought of as a *phenomenon of attraction* with respect to the total order related to activities (see Fig. 7). We point out, however, that certain intricacies of the general case do not occur in rank 3. In an arrangement of pseudolines a region is a polygon, hence, as in the uniform case, all its vertices are simple, i.e., incident to a number of facets equal to the dimension.

Example 4.1.1. Let D_{13} be the configuration of 13 points in the projective plane shown in Fig. 3. The configuration D_{13} is obtained by adding 3 points BCD to a

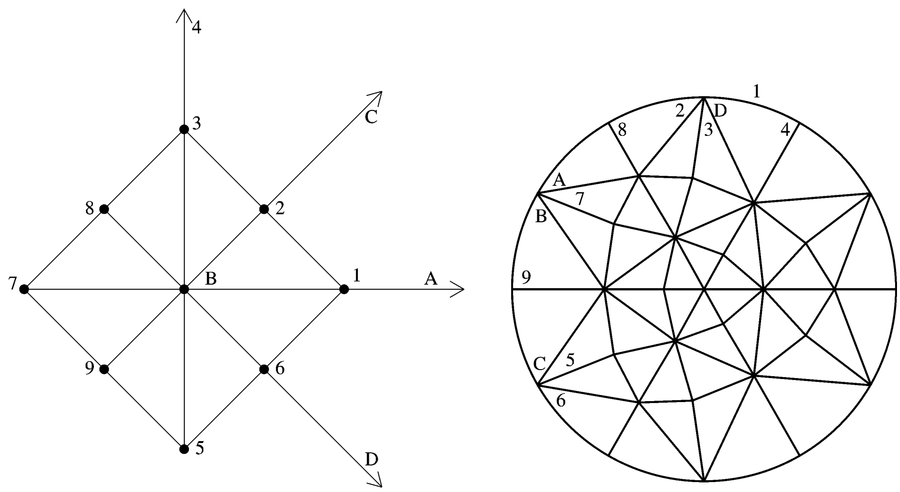


Fig. 3.

Desargue configuration on $123456789A$. Its automorphism group is of order 24, acting symmetrically on 1457, with 3 orbits, namely 1457 23689A BCD.

The pseudoline arrangement D_{13} contains all cases of Definition 4.1 below.

The Tutte polynomial of D_{13} is

$$t(D_{13}; x, y) = y^{10} + 3y^9 + 6y^8 + 10y^7 + 15y^6 + 21y^5 + 28y^4 + x^3 + 9xy^2 + 36y^3 + 10x^2 + 22xy + 36y^2 + 24x + 24y.$$

The matroid D_{13} has $t(D_{13}; 1, 1) = 246$ bases, and we have $b_{1,0} = 24$ $b_{2,0} = 10$ $b_{3,0} = 1$. The pseudoline arrangement of Fig. 3 has $24 + 2.10 + 4.1 = 48$ regions, with 24 bounded regions.

Definition 4.1. Let M be an ordered oriented matroid on a set $E = \{e_1 < e_2 < \cdots\}$. Without loss of generality, we may suppose that M has no 1- or 2-circuits (since a matroid with a loop has no $(1, 0)$-basis, and two parallel elements appear together and have the same sign in all cocircuits of an acyclic matroid). Let $B = \{e_1 < e_p < e_q\}$ be a $(1, 0)$-base of M.

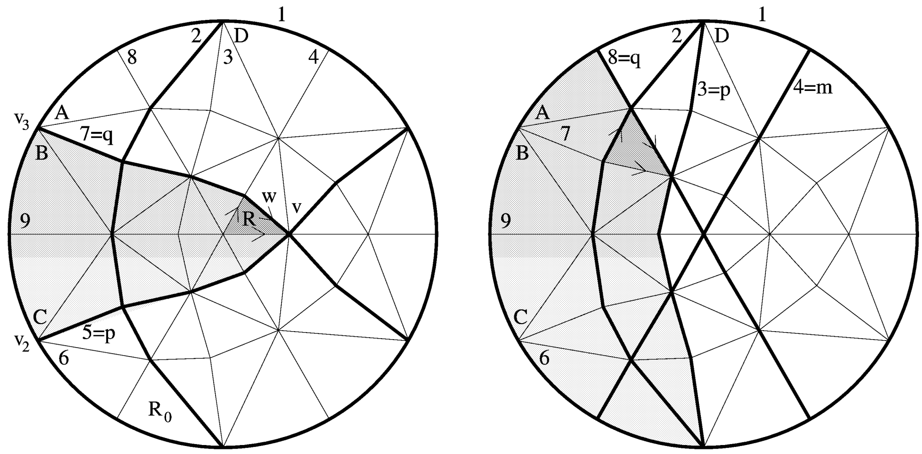
We have $e_p > e_2$, and e_p is the smallest pseudoline of M containing the intersection v of the pseudolines e_p and e_q (otherwise this smallest element e would be smallest in the circuit $\{e, e_p, e_q\}$, hence externally active with respect to B). In particular, e_2 does not contain v.

As in Section 3 we obtain the definition of the desired correspondence by applying Algorithm 3.0.1. There are four cases. We will give details for the first one, and leave the other three to the reader. In each case we define an *active quadrant Q*, intersection of 2 half-planes defined by e_p, e_q. Then *the region R associated with B by the active basis-reorientation correspondence is the region of the arrangement contained in Q, incident to the vertex $v = e_p \cap e_q$, and having one of its two edges incident to v supported by e_q.*

For short, we say that e_k, e_ℓ are *parallel* if $\{e_1, e_k, e_\ell\}$ is a circuit of M. We denote by e_m the smallest element $e_m > e_2$ which is not parallel to e_2. Then, $\{e_1, e_2, e_m\}$ is the lexicographically smallest basis.

(1) *Both e_p and e_q are not parallel to e_2* (Fig. 4, bases 147, 148, 149, 14A, 14C, 157, 158, 159, 168, 16B of Fig. 6).

By the hypothesis e_p e_q not parallel to e_2, we have $e_2 \in D_2 = C^*(B; e_p)$ and $e_2 \in D_3 = C^*(B; e_q)$. At the first step of the algorithm, we reorient $D_1 = C^*(B; e_1)$ positively. The region R is one of the regions incident to the vertex $v = v_1 = e_p \cap e_q$ corresponding to D_1. Second step: we reorient on $D_2 \setminus D_1$ so that after reorientation $D_2 = C^*(B; e_p)$ is positive on $D_2 \setminus D_1$ and has e_2 negative. The vertex $v_2 \in e_1 \cap e_q$ corresponding to D_2 is on the side of e_2 opposite to the side of R, therefore the edge w of the arrangement corresponding to the positive covector $D_1 \circ D_2$, which is the edge of e_q incident to $v = v_1$ directed toward v_2, is the edge of e_q incident to v directed toward $e_2 \cap e_q$. The region R is one of the 2 regions incident to the edge w. Third step: we reorient on $D_3 \setminus (D_1 \cup D_2)$ so that after reorientation $D_3 = C^*(B; e_q)$ is positive on $D_3 \setminus (D_1 \cup D_2)$ and has e_2 negative.



(1) B=157 (2a) 138

Fig. 4.

The vertex $v_3 \in e_1 \cap e_p$ corresponding to D_3 is on the side of e_2 opposite to the side of R. The region R corresponding to the positive covector $D_1 \circ D_2 \circ D_3$, contained in the side of e_q containing v_3, is now completely determined.

The active quadrant Q is the intersection of the closed halfplane defined by e_p and containing the intersection of e_2 and e_q, and the closed halfplane defined by e_q and containing the intersection of e_2 and e_p. The intersection of Q with e_2 is a bounded (pseudo)segment. Example—Fig. 4(1).

Let the fundamental region R_0 be the triangle with sides 1 2 4, and consider $B = 157$. We apply Algorithm 3.0.1. We have $D_1 = 1\bar{2}\bar{3}4\bar{6}\bar{8}A\bar{B}C$, $D_2 = \bar{2}\bar{3}456\bar{8}9C\bar{D}$, and $D_3 = 23\bar{4}789ABD$. First reorientation: $D_1^- = 2368B$. We get $D_2 = 2345\bar{6}89C\bar{D}$. Second reorientation: $D_2^+ \setminus D_1 = 59$. We get $D_3 = \bar{2}347\bar{8}\bar{9}A\bar{B}D$. Third reorientation: $D_3^- \setminus (D_1 \cup D_2)$ is empty. The reorientation associated with B is 235689B. It can easily be checked on Fig. 4(1) that the path 236B859 goes from the fundamental region to the shaded region associated with $B = 157$ by the above definition.

There is a degeneracy if at least one of e_p or e_q is parallel to e_2—then, exactly one, since $\{e_2, e_p, e_q\}$ is a basis. In this case, the definition of Q uses the pseudoline e_m. There may be two subcases, depending on whether v is contained in e_m or not.

(2a) *e_p parallel to e_2, v not contained in e_m* (Fig. 4, bases 136, 137, 138, 139, 13A, 13C of Fig. 6).

Then e_q is not parallel to e_2, and we have $e_q \neq e_m$ since $v \notin e_m$.

The active quadrant Q is the intersection of the closed halfplane defined by e_p containing the intersection of e_2 and e_q, and the closed halfplane defined by e_q containing the intersection of e_p and e_m.

(2b) *e_q parallel to e_2, v not contained in e_m* (Fig. 5, bases 15D, 16D of Fig. 6).

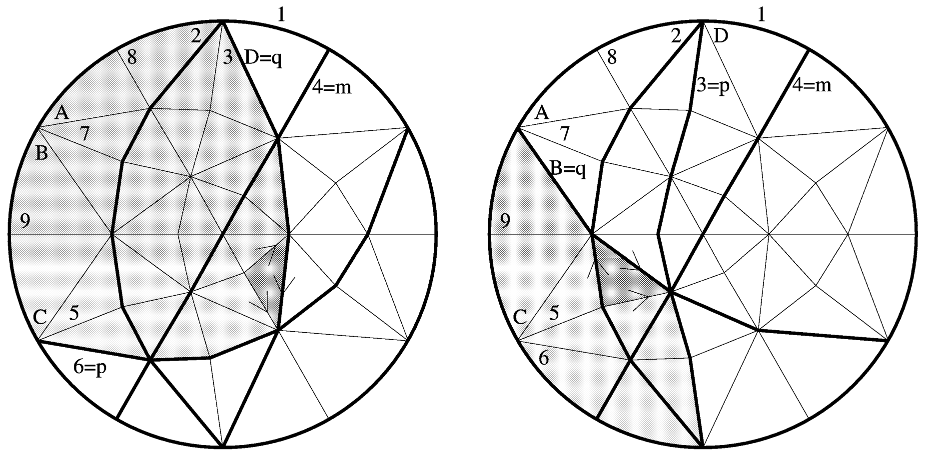


Fig. 5.

Then e_p is not parallel to e_2, and we have $e_p \neq e_m$ since $v \notin e_m$.

The active quadrant Q is the intersection of the closed halfplane defined by e_q containing the intersection of e_2 and e_p, and the closed halfplane defined by e_p containing the intersection of e_q and e_m.

(3) *e_p or e_q parallel to e_2, v contained in e_m* (Fig. 5, bases 135, 13B of Fig. 6).

If $v \in e_m$ and e_q parallel to e_2, then e_p is nonparallel to e_2, hence $m = p$ since p is the smallest pseudoline containing v, but then e_p would be internally active. Hence e_p is parallel to e_2 and e_q is not parallel to e_2, implying $e_q > e_m$ otherwise e_q would be internally active.

The active quadrant Q is the intersection of the closed halfplane defined by e_p containing the intersection of e_2 and e_q, and the closed halfplane defined by e_q containing the intersection of e_2 and e_m.

We point out that in Definition 4.1 two oriented matroid programs are used (see Section 3). In both the line at infinity is e_1. The first one has objective function e_2. When the set of solutions is 1-dimensional—the so-called degenerate case—a second program with objective function e_m is used to obtain a unique vertex.

Theorem 4.2. *The active basis-reorientation correspondence maps bijectively the set of $(1,0)$-bases onto the set of bounded regions of the pseudoline arrangement.*

Proof. We prove that the mapping is injective. Suppose there are two bases $B = \{e_1 < e_p < e_q\}$ and $B' = \{e_1 < e_{p'} < e_{q'}\}$ mapped to a same region R by the active basis-reorientation correspondence given by Definition 4.1.

In the case of a pseudoline arrangement, as already observed in Section 3, the cocircuit graph can be identified with the graph defined by the pseudolines. To obtain the active

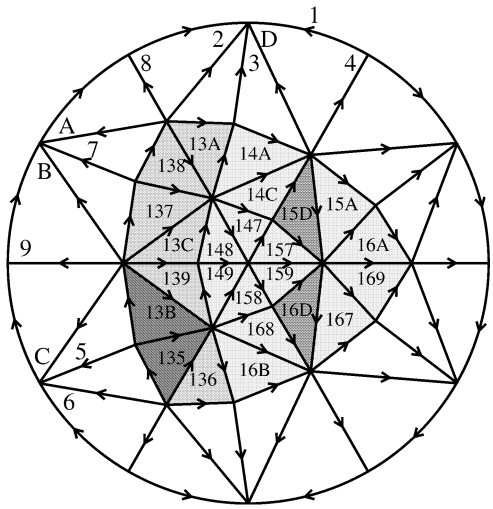


Fig. 6.

cocircuit graph, we direct the edges by means of Definition 3.6. Figure 6 shows the graph for D_{13} with all edge directions. To prove Theorem 4.2, it suffices to direct the finite edges, i.e., with no vertex on e_1: from e_2 towards e_1 for edges supported by pseudolines not parallel to e_2, from e_m towards e_1 for edges supported by pseudolines parallel to e_2.

In a bounded region R associated with a $(1,0)$-basis by the correspondence of Definition 4.1, the two edges incident to v are directed towards v. It follows easily from topological properties of pseudolines (the Jordan curve theorem) that all vertices of R different from v have outgoing edges. Hence, a region R, image of at least one basis determines the vertex v. It follows that e_p respectively e'_p is the smallest pseudoline containing v (otherwise this smallest pseudoline would be externally active with respect to B respectively B'). In particular, $e_p = e_{p'}$.

Suppose $e_q \neq e_{q'}$. Then the 2 edges of R incident to v are supported by e_q and $e_{q'}$. If both e_q and $e_{q'}$ are not parallel to e_2 then B and B' are both in one of the cases (1), (2a) or (3) of Definition 4.1. In case (1) cannot be of the same side than $e_2 \cap e_p$ for both e_q and $e_{q'}$. In case (2a) cannot be of the same side than $e_m \cap e_p$ for both e_q and $e_{q'}$. In case (3) cannot be of the same side than $e_m \cap e_2$ for both e_q and $e_{q'}$. If one of e_q, e'_q is parallel to e_2, say e_q, then B is in case (2b) and B' in case (1), and we have also an impossibility.

As in the proof of Theorem 3.3, injectivity implies bijectivity since $o_{1,0} = 2b_{1,0}$ [11]. □

Figure 6 illustrates the proof of Theorem 4.2. It shows edge directions in the active cocircuit graph. The shade of gray indicates the relevant case of Definition 4.1. The basis given by the active correspondence is written within each bounded region.

We complete Theorem 4.2 by proving directly the surjectivity of the correspondence. We need this proof to reverse locally the correspondence, i.e., to be able to write the basis associated with a bounded region of a pseudoline arrangement without computing the whole correspondence.

Lemma 4.2.1. *Every restriction of the active cocircuit graph to a region of the pseudoline arrangement has a unique sink.*

Proof. As already observed in Section 3, the bijectivity of the active correspondence on bounded regions implies the 'main theorem of oriented matroid programming,' i.e., the existence of a sink in all bounded regions in the nondegenerate case or of a 'sink edge' parallel to the pseudoline e_2 in the degenerate case.

Conversely, Lemma 4.2.1 can be obtained from oriented matroid programming. But a direct proof is an easy exercise on pseudoline arrangements. □

Proof of surjectivity. Let R be a bounded region of the pseudoline arrangement contained in e_1^+. We have do define a $(1,0)$-basis $B = \{e_1 < e_p < e_q\}$ such that the R is the image of B by the active basis-reorientation correspondence of Definition 4.1.

Let v be the sink of the restriction to R of the active cocircuit graph given by Lemma 4.2.1, $e < e'$ be the two edges of R incident to v. Necessarily the two pseudolines e_p and e_q contain v, the pseudoline e_p is smallest among the pseudolines containing v, and we have $e_q = e$ or $e_q = e'$.

If $e = e_p$, then necessarily $e' = e_q$. Suppose $e_p < e$. We distinguish several cases.

(a) *e_p is not parallel to e_2.*
  - (a1) If both e and e' are not parallel to e_2, let Q respectively Q' be the active quadrant defined by the pseudolines e_p and e respectively e' as in case (1) one Definition 4.1. Exactly one of Q or Q' contains R: we set $e_q = e$ if $R \subset Q$ respectively $e_q = e'$ if $R \subset Q'$.
  - (a2) If e respectively e' is parallel to e_2, setting $e_q = e$ respectively $e_q = e'$, we have case (2b) of Definition 4.1.

(b) *e_p is parallel to e_2.*

Then e and e' are not parallel to e_2. Let e_m be the smallest pseudoline not parallel to e_2.

(b1) If v is not in e_m, then e_q is defined as in (a1), with active quadrants defined by case (2a) of Definition 4.1.

(b2) If v is on e_m, then e_q is defined as in (a1), with active quadrants defined by case (3) of Definition 4.1. □

We complete the description of the canonical active basis-reorientation correspondence by considering internal bases of activities 2 and 3. As in Section 3 for the general uniform case, the construction is done by means of active partitions defined directly in each case. Up to parallel elements, the relevant minors, of rank $\leqslant 2$, are uniform, and results of Section 3 apply in very simple cases. We omit proofs. In each case, we indicate the relevant bases of D_{13} in Fig. 7. As in Definition 4.1, we denote by e_m the smallest pseudoline such that $\{e_1, e_2, e_m\}$ is not a circuit.

Definition 4.5. (1) $B = \{e_1 < e_2 < e_q\}$ (activity 2).

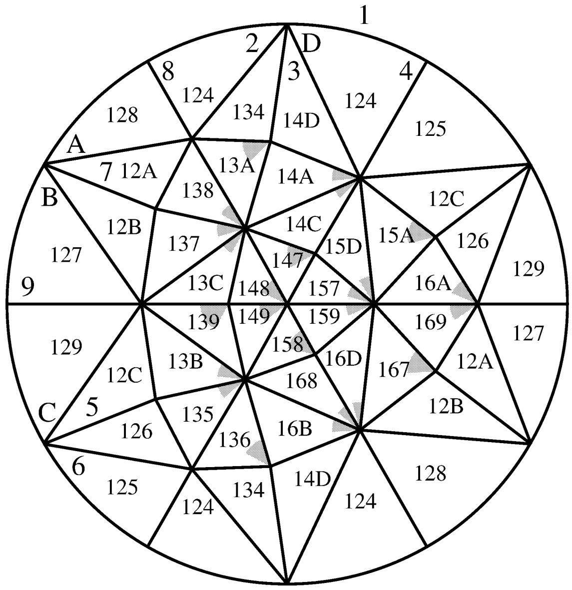


Fig. 7.

Let L be the set of pseudolines containing the intersection $\{v, v'\}$ of the pseudolines e_1 and e_q of B.

(1a) *e_q is the smallest element of $L \setminus \{e_1\}$* (bases 125, 127, 128, 129 of Fig. 7).

We have to consider M' obtained from M/e_1 by deleting all nonsmallest elements in each parallel class (the *active partition* is $E = \{e_1\} + \{e_2, e_3, \ldots\}$). This oriented matroid is uniform with rank 2.

In this case e_m does not contain v (otherwise $m = q$ and e_q is internally active). One region R is incident to v, bounded by a pseudosegment not meeting $e_1 \cap e_2$ with one extremity in $e_1 \cap e_q$ and the other in $e_1 \cap e_m$. The other region is $-_{E \setminus \{e_1\}} R$.

(1b) *The smallest element of $L \setminus \{e_1\}$ is e_p, and we have $e_p \neq e_q$* (bases 126, 12A, 12B, 12C of Fig. 7).

We have to consider $M' = M(L)$. The active partition is $E = L + E \setminus L$. This oriented matroid is uniform with rank 2. One region R is incident to v, bounded by e_q, and is contained in the side of e_q containing e_p. The other region is $-_{E \setminus L} R$.

(2) $B = \{e_1, e_m, e_q\}$ (activity 2) (bases 134, 14D of Fig. 7).

As in case (1b), the active partition is $E = L + (E \setminus L)$. One region R is incident to v, bounded by e_q, and contained in the side of e_q containing e_2. The other region is $-_{E \setminus L} R$.

(3) $B = \{e_1, e_2, e_m\}$ (activity 3) (base 124 of Fig. 7).

Let L be the set of pseudolines containing the intersection of the pseudolines e_1 and e_2. The active partition is $E = \{e_1\} + (L \setminus \{e_1\}) + (E \setminus L)$. The 4 regions associated with B in e_1^+ are those incident to $e_1 \cap e_2$ and bounded by e_1.

Figure 7 shows the canonical active basis-reorientation correspondence for internal bases and acyclic regions. The gray sector inside a bounded region indicates the vertex v of Definition 4.1 and the pseudoline e_q (which supports it, whereas the other edge of the region incident to v does not).

Theorem 4.6. *The canonical active basis-reorientation correspondence between the internal bases of an ordered oriented matroid of rank* 3 *and its acyclic reorientations has the required multiplicities.*

We omit the proof. We end this section by the counterpart of Proposition 3.9 for rank-3 matroids. Either by an easy direct proof, or by using the fact that a rank-3 oriented matroid is Euclidean [1, Chapter 10], it can be shown that the active cocircuit graph of a rank-3 oriented matroid has no directed cycles.

Proposition 4.7. *Let M be a rank-*3 *ordered oriented matroid on $E = \{e_1 < e_2 < \cdots\}$. The active basis-reorientation correspondence for $(1,0)$ activities is uniquely determined by the following two properties.*

(i) *The correspondence induces a bijection between $(1,0)$ bases and bounded regions of the pseudoline arrangement representing M.*
(ii) *Let $B = \{e_1 < e_p < e_q\}$ with $e_p > e_2$ be a $(1,0)$-basis, and R be the bounded region image of B. Then, the intersection of the pseudolines e_p and e_q is a vertex incident to R, and the pseudoline e_q supports an edge of R.*

Proof. The proof of Proposition 4.7 is similar to the proof of Proposition 3.9. □

In terms of programming, in the rank-3 acyclic case, the basis associated with a bounded region is the *optimal* basis for an extended linear program with respect to the total order. The element e_m is used to define the optimal vertex when e_2 does not suffice. Moreover, a basis defines two nested faces e_q and $e_p \cap e_q$ which have to be optimized, yielding a first example of flag matroid programming.

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